CREST and 4th Ritsumeikan-Florence Workshop on Risk Simulation and Related Topics

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Lectures on Credit Risk

1. Models for single default
2. Contagion models
3. Credit derivatives

Models for single default

1. Structural Approach
2. Hazard process Approach
3. Cox Processes
4. Density Models

Defaultable Claims

Let us first describe a generic defaultable claim:

1. **Default** of an entity occurs at time τ. Default may be bankruptcy or other financial distress.
2. At maturity T the **promised payoff** X is paid only if the default did not occurred.
3. The **promised dividends** A are paid up to default time.
4. The **recovery claim** $\widetilde{X}$ is received at time T, if default occurs prior to or at the claim's maturity date T.
5. The **recovery process** Z specifies the recovery payoff at time of default, if default occurs prior to or at the maturity date T.

Structural Approach

- A **risky asset** V, which may represent the value of the firm, is traded. The **riskless asset** (the savings account B) satisfies

$$dB_t = r_t \, B_t \, dt.$$

- The value of the firm V satisfies a Stochastic Differential Equation

- In the Black and Cox model, the default occurs at the first passage time of the value process V to a **deterministic** default-triggering barrier.
- More precisely, the default time equals

$$\tau = \inf \{ t \in [0, T] : V_t < v(t)\}$$

for some function v.

Corporate Bond

A **corporate bond** is defined as the following defaultable claim

$$X = L,\ A = 0,\ Z = \beta_2 V,\ \widetilde{X} = \beta_1 V_T,$$

where β_1, β_2 are constants in $[0, 1]$.

If V is continuous, the default time τ is **predictable**: there exists a sequence of stopping times τ_n such that

$$\tau_n < \tau, \quad \tau_n \to \tau$$

The price of a defaultable bond with $Z = \widetilde{X} = 0$ goes to 0, when t is closed to τ.

It is usually difficult to obtain the law of the first hitting time. Computations can be done for a geometric Brownian motion and a constant barrier.

In case of discontinuous processes, the computation is even more difficult. Results are obtained for a double exponential compound process. In Kou's model, the barrier is constant and

$$V_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \,,$$

where the density of the law of Y_1 is

$$\nu(dx) = \big(p\eta_1 e^{-\eta_1 x} 1\!\!1_{\{x>0\}} + (1-p)\eta_2 e^{\eta_2 x} I_{\{x<0\}}\big)\, dx \,.$$

In this case, the default time τ is **totally inaccessible**: $\mathbb{P}(\tau = \vartheta) = 0$ for any predictable stopping time ϑ

Shortcomings of Structural Approach

1. Assumes the total value of firm assets can be easily observed.
2. Postulates that the total value of firm assets is a tradable security.
3. Generates low credit spreads for corporate bonds close to maturity.
4. Requires a judicious specification of the default barrier in order to get a good fit with the observed spread curves.
5. Defaults can be determined by factors other than assets and liabilities (for example, defaults could occur for reasons of illiquidity).

Further Developments

Incomplete /noisy information

Delay

Hazard process Approach

1. General case
2. $(\mathcal{H})$-Hypothesis
3. Representation theorem
4. Intensity approach

General case

The model

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We denote by $\mathcal{G}_t \stackrel{def}{=} \mathcal{F}_t \vee \mathcal{H}_t$. If $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

Therefore any $\mathcal{G}_t$-measurable random variable Y_t satisfies
$1\!\!1_{\{\tau > t\}} Y_t = 1\!\!1_{\{\tau > t\}} y_t$, **where y_t is an $\mathcal{F}_t$-measurable random variable**.

Key lemma

We denote by $F_t \stackrel{def}{=} \mathbb{P}(\tau \leq t|\mathcal{F}_t)$ the conditional law of τ given the information $\mathcal{F}_t$, and $G_t = 1 - F_t$. Note that G is a **supermartingale**. **We assume that G is continuous and $G_t > 0$.**

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Let X be an $\mathcal{F}_T$-measurable integrable r.v. Then,

$$\mathbb{E}(X \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}(X \mathbb{1}_{\{\tau>T\}} | \mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}} | \mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}(X e^{-\Gamma_T} | \mathcal{F}_t).$$

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Let h be an $\mathbb{F}$-predictable process. Then,

$$\mathbb{E}(h_\tau 1\!\!1_{\tau<T} | \mathcal{G}_t) = h_\tau 1\!\!1_{\{\tau \leq t\}} + 1\!\!1_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}\left(\int_t^T h_u dF_u | \mathcal{F}_t \right) .$$

Martingales

(i) The process $L_t \stackrel{def}{=} (1 - H_t)e^{\Gamma_t}$ *is a* $\mathbb{G}$-***martingale***.

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$$M_t \stackrel{def}{=} H_t - \int_0^{t\wedge\tau} \frac{dA_u}{G_u}$$

is a $\mathbb{G}$*-martingale.*

Proofs: *The process* $L_t = (1 - H_t)e^{\Gamma_t}$ *is a* $\mathbb{G}$-***martingale***.

From the key lemma, for $t > s$

$$\mathbb{E}(L_t|\mathcal{G}_s) = \mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{F}_s)$$

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$$M_t \stackrel{def}{=} \int_{]0,t]} dH_u - \int_{]0,t]} (1 - H_u)d\Gamma_u = -\int_{]0,t]} e^{-\Gamma_u} dL_u$$

and is a $\mathbb{G}$-martingale since L is $\mathbb{G}$-martingale.

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Let $s < t$. We give the proof in two steps, using the Doob-Meyer decomposition of F as $F_t = Z_t + A_t$.

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&= 1\!\!1_{\tau\leq s} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)
\end{aligned}$$

In a second step, we prove that, setting $\Lambda_u = \int_0^u \frac{dA_s}{G_s}$,

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\end{aligned}$$

We now use IP formula, using that Λ is bounded variation and continuous

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hence

$$\begin{aligned} \int_s^t \Lambda_u dF_u + \Lambda_t G_t &= -\Lambda_t G_t + \Lambda_s G_s + A_t - A_s + \Lambda_t G_t \\ &= \Lambda_s G_s + A_t - A_s \end{aligned}$$

From

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From

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

and

$$\mathbb{E}(\Lambda_{t\wedge\tau}|\mathcal{G}_s) = \Lambda_{s\wedge\tau} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(A_t - A_s|\mathcal{F}_s\right)$$

we deduce

$$\mathbb{E}(H_t - \Lambda_{t\wedge\tau}|\mathcal{G}_s) = H_s - \Lambda_{s\wedge\tau}$$

The continuity of A is equivalent to the fact that τ is totally inaccessible.

Moreover, if A is absolutely continuous w.r.t. the Lebesgue measure, there exists an $\mathbb{F}$-adapted process λ, called the $\mathbb{F}$-intensity such that the process

$$H_t - \int_0^{t\wedge\tau} \lambda_u du = H_t - \int_0^t (1 - H_u)\lambda_u du$$

is a $\mathbb{G}$-martingale. The process λ satisfies

$$\lambda_t = \lim_{h\to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t + h|\mathcal{F}_t)}{\mathbb{P}(t < \tau|\mathcal{F}_t)}$$

The $\mathbb{G}$-intensity of τ is the $\mathbb{G}$ adapted process $\lambda^{\mathbb{G}}$ such that the process $H_t - \int_0^t \lambda^{\mathbb{G}}_u du$ is a $\mathbb{G}$-martingale. Obviously $\lambda^{\mathbb{G}}_t = 1\!\!1_{t<\tau}\lambda_t$.

Computation in a restricted filtration

Let $\widetilde{\mathbb{F}} \subset \mathbb{F}$ and $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$.

From

$$F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$$

we deduce

$$\widetilde{F}_t = \mathbb{P}(\tau \leq t | \widetilde{\mathcal{F}}_t) = \mathbb{E}(F_t | \widetilde{\mathcal{F}}_t)$$

The computation of the intensity is more difficult, the $\widetilde{\mathbb{F}}$- intensity in the restricted filtration is not the conditional expectation of the $\mathbb{F}$-intensity

This methodology provides a bridge between structural approach and reduced one: starting with a structural model, where $\tau = \inf\{t : X_t \leq 0\}$ where the reference filtration is the one generated by X, restricting the information to $\widetilde{\mathbb{F}}$, and/or adding noise to the reference filtration leads to a reduced form model where the goal is to compute the conditional law of τ given the information, or at least the quantity $\mathbb{P}(\tau > t|\widetilde{\mathcal{F}}_t)$.

($\mathcal{H}$) Hypothesis

We now examine the **immersion property** (or ($\mathcal{H}$)-hypothesis) which reads:

($\mathcal{H}$) **Every $\mathbb{F}$ square-integrable martingale is a $\mathbb{G}$ square integrable martingale.**

This hypothesis implies that the $\mathbb{F}$-Brownian motion remains a Brownian motion in the enlarged filtration and that every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale .

This is equivalent to : For any $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_\infty).$$

Complete model case

We assume constant interest rate. Let S be a semi-martingale on $(\Omega, \mathcal{G}, \mathbb{P})$ such that there exists a **unique** probability $\mathbb{Q}$, equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, where $\mathcal{F}_t = \mathcal{F}_t^S = \sigma(S_s, s \leq t)$ such that $(\widetilde{S}_t = S_t e^{-rt}, 0 \leq t \leq T)$ is an $\mathbb{F}^S$-martingale under the probability $\mathbb{Q}$.
We assume that there exists a probability $\tilde{\mathbb{Q}}$, equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ such that $(\widetilde{S}_t, 0 \leq t \leq T)$ is a $\mathbb{G}$-martingale under the probability $\tilde{\mathbb{Q}}$.
Then, **any $(\mathbb{F}, \mathbb{Q})$-martingale is a $(\mathbb{G}, \tilde{\mathbb{Q}})$-martingale** and the restriction of $\tilde{\mathbb{Q}}$ to $\mathcal{F}_T$ is equal to $\mathbb{Q}$.

Change of a probability measure

Kusuoka shows, by means of a counter-example, that the hypothesis ($\mathcal{H}$) is not invariant with respect to an equivalent change of the underlying probability measure, in general.

Representation theorem

Kusuoka establishes the following representation theorem, in the case where $\mathbb{F}$ is a Brownian filtration. Under $(\mathcal{H})$, any $\mathbb{G}$-square integrable martingale admits a representation as the sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M.

Suppose that hypothesis $(\mathcal{H})$ holds under $\mathbb{P}$ and that any $\mathbb{F}$-martingale is continuous. Then, the martingale $M_t^h = \mathbb{E}_{\mathbb{P}}(h_\tau \,|\, \mathcal{G}_t)$, where h is an $\mathbb{F}$-predictable process such that $\mathbb{E}(h_\tau) < \infty$, admits the following decomposition

$$M_t^h = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u^h + \int_{]0,t\wedge\tau]} (h_u - J_u)\, dM_u,$$

where m^h is the continuous $\mathbb{F}$-martingale

$$m_t^h = \mathbb{E}_{\mathbb{P}}\Big(\int_0^\infty h_u dF_u \,|\, \mathcal{F}_t \Big),$$

$J_t = e^{\Gamma_t}(m_t^h - \int_0^t h_u dF_u)$ is the predefault value of h_τ and M is the discontinuous $\mathbb{G}$-martingale $M_t = H_t - \Gamma_{t\wedge\tau}$.

PROOF: : We know that

$$\begin{aligned} M_t^h &= \mathbb{E}(h_\tau | \mathcal{G}_t) \\ &= \mathbb{1}_{\{\tau \le t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}\Big(\int_t^\infty h_u dF_u \Big| \mathcal{F}_t \Big) \end{aligned}$$

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If $\mathbb{F}$ is a Brownian filtration, $dm_u^h = \sigma_u^h dW_u$

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\end{aligned}$$

From the facts that Γ is an increasing process

m^h a continuous martingale

and using the integration by parts formula, we deduce that

$$dJ_t = e^{\Gamma_t} dm_t^h + (J_t - h_t) \frac{dF_t}{G_t}$$

This theorem generalizes : if W is a $\mathbb{F}$-Brownian motion which decomposes as

$$W_t = B_t + \int_0^t \mu_s ds$$

where B is a $\mathbb{G}$-martingale (and a $\mathbb{G}$ Brownian motion), then, any $\mathbb{G}$ martingale admits a decomposition as

$$Y_t = y + \int_0^t \widehat{y}_s dB_s + \int_0^t \widetilde{y}_s dM_s$$

Intensity approach

In the so-called intensity approach, the default time τ is a $\mathbb{G}$-stopping time. The intensity is defined as any non-negative process λ, such that

$$M_t \stackrel{def}{=} H_t - \int_0^{t \wedge \tau} \lambda_s ds$$

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The existence of the intensity relies on the fact that H is a sub-martingale and can be written as $M + A$ where M is a martingale M and A a predictable increasing process. The increasing process A is such that $A_t 1\!\!1_{t\geq\tau} = A_\tau 1\!\!1_{t\geq\tau}$.

The intensity exists only if τ is a totally inaccessible stopping time.

We emphasize that, in that setting the intensity is not well defined after time τ, i.e., if λ is an intensity, for any non-negative predictable process g the process $\tilde{\lambda}_t = \lambda_t 1\!\!1_{t\leq\tau} + g_t 1\!\!1_{\{t>\tau\}}$ is also an intensity.

If the process $Y_t = \mathbb{E}\left(X \exp\left(-\int_0^T \lambda_u du\right) |\mathcal{G}_t\right)$ ***is continuous at time*** τ, *then, setting* $L_t = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}$

$$\mathbb{E}(X \mathbb{1}_{\{T<\tau\}} | \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \mathbb{E}\left(X \exp\left(-\int_t^T \lambda_u du\right) |\mathcal{G}_t\right) = L_t Y_t$$

If Y *is not continuous*

$$\mathbb{E}(X \mathbb{1}_{\{T<\tau\}} | \mathcal{G}_t) = L_t Y_t - \mathbb{E}(\Delta Y_\tau \mathbb{1}_{\tau<T} | \mathcal{G}_t) \, .$$

It can be mentioned that the continuity of the process depends on the choice of λ after time τ.

Proof: Setting

$$U_t = L_t Y_t = 1\!\!1_{t<\tau} \exp\left(\int_0^t \lambda_s ds\right) \mathbb{E}\left(X \exp\left(-\int_0^T \lambda_u du\right) |\mathcal{G}_t\right)$$

we have $U_T = X 1\!\!1_{\{T<\tau\}}$ and

$$dU_t = L_{t-} dY_t + Y_{t-} dL_t + d[L, Y]_t = L_{t-} dY_t + Y_{t-} dL_t + \Delta L_t \Delta Y_t$$

and

$$\mathbb{E}(U_T|\mathcal{G}_t) = \mathbb{E}(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = U_t - \mathbb{E}(e^{\Lambda_\tau} \Delta Y_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) \,.$$

Then, for any $X \in \mathcal{G}_T$:

$$\mathbb{E}(X \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{\tau>t} \left(e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T} X | \mathcal{G}_t) - \mathbb{E}(e^{\Lambda_\tau} \Delta Y_\tau \mathbb{1}_{\tau<T} | \mathcal{G}_t) \right)$$

where $Y_t = \mathbb{E}\left(X \exp\left(-\Lambda_T\right) | \mathcal{G}_t\right)$ and $\Lambda_t = \int_0^t \lambda_u du$

Cox Processes and Extensions

1. Construction of Default Time with a given Intensity
2. Properties
   - 2.1 Conditional expectation
   - 2.2. Key Lemma
3. Defaultable Assets

Default Time with a given Intensity

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F}$.

A nonnegative $\mathbb{F}$-adapted process λ is given.

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We define the random time τ as the first time when the process $\Lambda_t = \int_0^t \lambda_s \, ds$ is above the random level Θ, i.e.,

$$\tau = \inf \{t \geq 0 \, : \, \Lambda_t \geq \Theta\}.$$

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$$\tau = \inf \{t \geq 0 \, : \, \Lambda_t \geq \Theta\}.$$

In particular, $\{\tau > s\} = \{\Lambda_s < \Theta\}$.

Properties

Conditional Expectations

The conditional distribution function of τ *given the* σ*-field* $\mathcal{F}_t$ *is for* $t \geq s$

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp\left(-\Lambda_s\right).$$

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Conditional Expectations

The conditional distribution function of τ given the σ-field $\mathcal{F}_t$ is for $t \geq s$

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp\left(-\Lambda_s\right).$$

Proof : For $s \leq t$,

$$\begin{aligned}\mathbb{P}(\tau > s|\mathcal{F}_t) &= \mathbb{P}(\Lambda_s < \Theta|\mathcal{F}_t) \\ &= \Psi(\Lambda_s)\end{aligned}$$

where $\Psi(x) = \mathbb{P}(x < \Theta)$.

Key lemma

Let Y be an integrable r.v. Then,

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}(Y|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}(Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}}e^{\Lambda_t}\mathbb{E}(Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t).$$

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If $X \in \mathcal{F}_T$

$$\mathbb{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, e^{\Lambda_t}\mathbb{E}(Xe^{-\Lambda_T}|\mathcal{F}_t).$$

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$$\boxed{\mathbb{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, e^{\Lambda_t}\mathbb{E}(Xe^{-\Lambda_T}|\mathcal{F}_t).}$$

The process λ is called **the intensity** of τ.

In particular, one an check that

(i) The process $L_t = 1\!\!1_{t<\tau} e^{\Lambda_t} = (1 - H_t) e^{\Lambda_t}$ is a martingale

(ii) Let X be an $\mathcal{F}_\infty$-measurable r.v.. Then

$$\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t)\,.$$

Stochastic Barrier

Suppose that

$$P(\tau \leq t | \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing $\mathbb{F}$-adapted process. Then, $(\mathcal{H})$ holds. Moreover, there exists a random variable Θ, independent of $\mathcal{F}_\infty$, with exponential law of parameter 1, such that $\tau \stackrel{law}{=} \inf\{t \geq 0 : \Gamma_t > \Theta\}$. In fact $\Theta \stackrel{def}{=} \Gamma_\tau$.

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where Γ is an arbitrary continuous strictly increasing $\mathbb{F}$-adapted process. Let us set $\Theta \stackrel{def}{=} \Gamma_\tau$. Then

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

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$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$. Therefore

$$P(\Theta > u | \mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability law of Θ and its independence of the σ-field $\mathcal{F}_\infty$. Furthermore,
$\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$.

We now compute the expectation of the value at time τ of a predictable process.

(i) If h is an $\mathbb{F}$-predictable (bounded) process then

$$\mathbb{E}(h_\tau|\mathcal{G}_t) = e^{\Lambda_t}\mathbb{E}\Big(\int_t^\infty h_u dF_u \,\Big|\, \mathcal{F}_t\Big)\mathbb{1}_{\{\tau>t\}} + h_\tau \mathbb{1}_{\{\tau\le t\}}$$

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$$\mathbb{E}(h_\tau) = \mathbb{E}\Big(\int_0^\infty h_u \lambda_u \exp\big(-\Lambda_u\big)\, du\Big)$$

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(ii) The process $(M_t := H_t - \int_0^{t\wedge\tau}\lambda_s ds, t\ge 0)$ is a $\mathbb{G}$-martingale.

(iii) The martingale $L_t = \mathbb{1}_{t<\tau}e^{\Lambda_t}$ satisfies $dL_t = -L_{t-}dM_t$.